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Unbiased Monte Carlo Evaluation of Certain Functional Integrals

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A Monte Carlo algorithm for the evaluation of certain functional integrals is introduced. This algorithm is not based on the approximation of the functional integral by a finite-dimensional integral. Unbiased estimators for the functional integrals are constructed and corresponding variance reduction techniques are considered. Numerical results of simulation studies are presented.

1. INTRODUCTION

Integration in function spaces is a powerful tool in many fields of mathematics and physics (cf. [10, 21, 26, 2, 1, 4]). Classical papers by Kac [17, 18] dealt with the probability distribution of certain functionals of the Wiener process. Connections with the solution of a corresponding partial differential equation were found. For example, the solution of the generalized heat flow equation

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) + c(x)\, u(t, x), \qquad u(0, x) = 1,$$

can be represented via the Feynman–Kac formula

$$u(t, x) = E \exp\left(\int_0^t c(x + w(s))\, ds \right) = \int_C F(y)\, d\mu_w(y), \tag{1.1}$$

where $(t, x) \in [0, \infty) \times R$, w is the standard Wiener process, μ_w denotes the Wiener measure on the space C of continuous functions on $[0, t]$, and $F(y) := \exp(\int_0^t c(x + y(s))\, ds)$.

Numerical methods for Wiener integrals and conditional Wiener integrals (related to the Brownian bridge) have been investigated in many papers (cf. [6, 14, 15, 19, 13, 8, 25, 3, 5]). All these methods are based on some approximation of the

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functional integral by an integral over a finite-dimensional space. For instance, Chorin [8] constructed remarkably simple approximation formulas of the form

$$\int_C F(y)\, d\mu_w(y) = (2\pi)^{-n/2} \int_{R^n} F_n(u_1, \ldots, u_n) \times \exp(-(u_1^2 + \cdots + u_n^2)/2)\, du_1 \cdots du_n + O(n^{-2}).$$

The implementation of such numerical methods is connected with the evaluation of high-dimensional integrals. This is a field in which the Monte Carlo method can compete successfully with usual deterministic quadrature formulas. Gel'fand *et al.* [14, 15] were apparently the first to deal with the application of the Monte Carlo method to the evaluation of functional integrals, although a "suggestion to apply the Monte Carlo method" had already been made in [7]. Further work in this direction has been done in [19, 8, 9, 20, 25, 5].

In this paper we present a Monte Carlo approach to the evaluation of functional integrals that is not based on their previous approximation by finite-dimensional integrals. Namely, we propose unbiased estimators for functional integrals of the form

$$E \exp\left(\int_0^T c(s, X(s))\, ds\right), \tag{1.2}$$

where X is a Markov process in the d-dimensional Euclidean space R^d with the initial condition $X(0) = 0$. The transition function $P_{tr}(t, x, s, A)$, $0 \leqslant t \leqslant s \leqslant T < \infty$, $x \in R^d$, is assumed to be known explicitly and to be measurable in (t, x, s), for any Borel set $A \subset R^d$. The function c is supposed to be real-valued and measurable on $[0, T] \times R^d$.

The application of the Monte Carlo method to the evaluation of functional integrals is usually connected to two main kinds of error. These are, on the one side, the systematic error resulting from the approximation of the functional integral by a finite-dimensional integral, and, on the other side, the statistical error arising from the evaluation of the mean value of the estimator for the finite-dimensional integral by taking the average over independent samples. Simulation studies in the literature (cf. [15, 19, 25]) are to illustrate the convergence of the systematic error. But it is very difficult to separate the systematic error from the statistical error and to answer the question whether it is small enough, especially in real situations, when the exact value of the functional integral is not known in advance. Using unbiased estimators we avoid this problem since only the statistical error is involved. The statistical error can be estimated by means of confidence intervals during the process of computation.

Our approach is based on a transformation of the functional integral into an integral over a countable union of finite-dimensional spaces. This transformation is performed in Section 2 (Proposition 1). Unbiased estimators for the functional integrals (1.2) are constructed in Section 3 (Proposition 3). The corresponding probability measures can be generated by means of appropriately chosen Markov

chains. The relation of these estimators to the von Neumann–Ulam scheme for solving linear integral equations is discussed. Some variance reduction techniques are considered in Section 4. Results of simulation studies are presented in Section 5. In Section 6, we summarize some advantages of our unbiased estimation scheme. Finally, we discuss possible generalizations and directions for further study.

2. Basic Properties

Let $X_{t,x}(u)$ $(0 \leqslant t \leqslant u \leqslant T < \infty,\ x \in R^d,\ X_{t,x}(t) = x)$ be a Markov family with the state space R^d and the transition function $P_{\mathrm{tr}}(t, x, u, dy)$. Further let c be a real-valued and measurable function on $[0, T] \times R^d$. We consider the functionals

$$I(t, x) := E \exp\left(\int_t^T c(u, X_{t,x}(u))\, du\right) \tag{2.1}$$

and

$$I_a(t, x) := E \exp\left(\int_t^T |c(u, X_{t,x}(u))|\, du\right),\ (t, x) \in [0, T] \times R^d,$$

taking values from $[0, +\infty]$. Further, we introduce the space $Y = \bigcup_{n=0}^{\infty} Y_n$, where $Y_n = ([0, T] \times R^d)^{n+1}$, $n \geqslant 0$. The following proposition plays a major role in the construction of unbiased estimators for the functional integrals (2.1).

Proposition 1. *Let $(t, x) \in [0, T] \times R^d$ be fixed and suppose that $I_a(t, x) < \infty$. Then*

(i) *a finite signed measure $v_{t,x}$ on Y can be defined via the formulas*

$$\begin{aligned} v_{t,x}|_{Y_n}(dt_0, dx_0; \ldots; dt_n, dx_n) &= \delta_{t,x}(dt_0, dx_0) \\ &\chi_{[t_0, T]}(t_1)\, c(t_1, x_1)\, P_{\mathrm{tr}}(t_0, x_0, t_1, dx_1)\, dt_1 \\ &\times \cdots \times \chi_{[t_{n-1}, T]}(t_n)\, c(t_n, x_n)\, P_{\mathrm{tr}}(t_{n-1}, x_{n-1}, t_n, dx_n)\, dt_n, \end{aligned}$$

where $v_{t,x}|_{Y_n}$ denotes the restriction of $v_{t,x}$ to Y_n, $n \geqslant 0$, $\delta_{t,x}$—the measure concentrated at the point (t, x), and χ_A—the indicator of a set A;

(ii) $I(t, x) = v_{t,x}(Y)$;

(iii) $I_a(t, x) = |v_{t,x}|(Y)$, *where $|v_{t,x}|$ denotes the total variation of the measure $v_{t,x}$.*

Proof. First we expand the exponent of (2.1) in a power series. Under the assumption $I_a(t, x) < \infty$ one can change the order of summation and integration so that

$$I(t, x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} E\left(\int_t^T c(u, X_{t,x}(u))\, du\right)^n.$$

Using simple symmetry arguments and expressing the finite-dimensional distributions of $X_{t,x}$ via the transition function, we obtain

$$E\left(\int_t^T c(u, X_{t,x}(u))\,du\right)^n = n!\int_t^T\int_{t_1}^T\cdots\int_{t_{n-1}}^T E(c(t_1, X_{t,x}(t_1))$$
$$\times\cdots\times c(t_n, X_{t,x}(t_n)))\,dt_n\cdots dt_1$$
$$= n!\int_t^T\int_{t_1}^T\cdots\int_{t_{n-1}}^T\int_{R^d}\cdots\int_{R^d}$$
$$\times c(t_1, x_1)\,P_{tr}(t, x, t_1, dx_1)$$
$$\times\cdots\times c(t_n, x_n)\,P_{tr}(t_{n-1}, x_{n-1}, t_n, dx_n)\,dt_n\cdots dt_1.$$

This yields (i) and (ii). Finally, we obtain (iii) from (ii) replacing the function c by its absolute value. This completes the proof.

The following proposition states some auxiliary results concerning the functionals $I(t, x)$ and $I_a(t, x)$.

PROPOSITION 2. *Let $(t, x)\in[0, T]\times R^d$ be fixed and suppose that $I_a(t, x)<\infty$. Then*

(i) $I(t, x) = v_{t,x}(\bigcup_{n=0}^{k-1} Y_n) + \int_{Y_k} I(t_k, x_k)$

$$dv_{t,x}((t_0, x_0;\ldots; t_k, x_k)) \qquad \textit{for any } k = 1, 2,\ldots;$$

(ii) $I_a(t, x) = |v_{t,x}|(\bigcup_{n=0}^{k-1} Y_n) + \int_{Y_k} I_a(t_k, x_k)$

$$d|v_{t,x}|((t_0, x_0;\ldots; t_k, x_k)), \qquad \textit{for any} \quad k = 1, 2,\ldots;$$

(iii) $|v_{t,x}|(\{(t_0, x_0;\ldots; t_n, x_n)\in Y_n: I_a(t_k, x_k) = \infty\}) = 0$, *for any* $0\leqslant k\leqslant n$.

Proof. It follows from Proposition 1(iii) that

$$I_a(t, x) = |v_{t,x}|\left(\bigcup_{n=0}^{k-1} Y_n\right) + \sum_{n=k}^{\infty} |v_{t,x}|(Y_n),$$

for any $k = 1, 2,\ldots$. The second term can be transformed into

$$\int_{Y_k} d\,|v_{t,x}|((t_0, x_0;\ldots; t_k, x_k))\left[\sum_{n=0}^{\infty}\int_{Y_n}\delta_{t_k,x_k}(du_0, dy_0)\right.$$
$$\times\chi_{[u_0, T]}(u_1)\,|c(u_1, y_1)|\,P_{\mathrm{tr}}(u_0, y_0, u_1, dy_1)\,du_1$$
$$\left.\times\cdots\times\chi_{[u_{n-1}, T]}(u_n)\,|c(u_n, y_n)|\,P_{\mathrm{tr}}(u_{n-1}, y_{n-1}, u_n, dy_n)\,du_n\right]$$

and is finite because of the condition $I_a(t, x) < \infty$. Consequently, the term in brackets is finite almost surely with respect to $|v_{t,x}|$ on Y_k, and can be replaced by $I_a(t_k, x_k)$ according to Proposition 1(iii). Thus, assertion (ii) follows. To prove (i), one proceeds in a similar way using Proposition 1(ii) and the fact that $I_a(t, x)$ is a majorant for $I(t, x)$.

Now, let $0 \leqslant k \leqslant n$ be fixed. We can write

$$
\begin{aligned}
&|v_{t,x}|(\{(t_0, x_0; \ldots; t_n, x_n) \in Y_n : I_a(t_k, x_k) = \infty\}) \\
&\quad = \int_{Z_k} d|v_{t,x}|((t_0, x_0; \ldots; t_k, x_k)) \left[\int_{Y_{n-k}} \delta_{t_k, x_k}(du_0, dy_0) \right. \\
&\quad \times \chi_{[u_0, T]}(u_1) |c(u_1, y_1)| P_{\mathrm{tr}}(u_0, y_0, u_1, dy_1)\, du_1 \\
&\quad \times \cdots \times \chi_{[u_{n-k-1}, T]}(u_{n-k}) |c(u_{n-k}, y_{n-k})| \\
&\quad \left. \times P_{\mathrm{tr}}(u_{n-k-1}, y_{n-k-1}, u_{n-k}, dy_{n-k})\, du_{n-k} \right],
\end{aligned}
$$

where $Z_k = \{(t_0, x_0; \ldots; t_k, x_k) \in Y_k : I_a(t_k, x_k) = \infty\}$. The function in brackets is integrable on Y_k since

$$
\int_{Y_k} [\cdots]\, d|v_{t,x}|((t_0, x_0; \ldots; t_k, x_k)) = |v_{t,x}|(Y_n) < \infty,
$$

because of $I_a(t, x) < \infty$. Further, we observe from (ii) that $|v_{t,x}|(Z_k) = 0$. Thus, assertion (iii) follows, which completes the proof.

3. Unbiased Estimators

We denote, for simplicity, $X := X_{0,0}$, $v := v_{0,0}$, and assume

$$
I_a(0, 0) = E \exp\left(\int_0^T |c(u, X(u))|\, du \right) < \infty. \tag{3.1}
$$

Proposition 1 enables us to construct a large variety of unbiased estimators for the functional integral $I(0, 0) = E \exp(\int_0^T c(u, X(u))\, du)$. The following assertions are immediate consequences of Proposition 1 and well-known results from the Monte Carlo integration theory (cf. [11, Chap. 4] or [24, Chap. 4]).

Proposition 3. *Let μ be a probability measure on Y such that the Radon–Nikodym derivative $dv/d\mu$ exists.* (3.2)

Let the estimator ξ be defined on (Y, μ) via

$$
\xi(y) = \frac{dv}{d\mu}(y). \tag{3.3}
$$

Then

(i) $E\xi := \int_Y \xi(y)\, d\mu(y) = I(0, 0)$,

(ii) $E|\xi| = I_a(0, 0)$,

(iii) $D\xi := E\xi^2 - (E\xi)^2 \geqslant I_a(0, 0)^2 - I(0, 0)^2$,

(iv) $D\xi = I_a(0, 0)^2 - I(0, 0)^2$ iff $\mu = |v|/I_a(0, 0)$.

A suitable way to define probability measures on Y is to use a Markov chain with an absorbing state. Let the chain start in $(0, 0)$ at time zero. Consider now the chain being in (t_k, x_k) at time k. Then it either will be absorbed (i.e., pass to the absorbing state) with probability $p_0(t_k, x_k)$ or will pass to the next state (t_{k+1}, x_{k+1}) according to a transition function $P(t_k, x_k; dt_{k+1}, dx_{k+1})$. This Markov chain has paths of the form $(0, 0; t_1, x_1; \ldots; t_n, x_n)$, where n is the random absorption time, which we also call the length of the path. We assume that this random length of the trajectories is a.s. finite. Then the Markov chain generates the following probability measure μ on Y,

$$\begin{aligned}\mu|_{Y_n}(dt_0, dx_0; \ldots; dt_n, dx_n) &= \delta_{0,0}(dt_0, dx_0)(1 - p_0(t_0, x_0)) \\ &\times P(t_0, x_0; dt_1, dx_1) \cdots (1 - p_0(t_{n-1}, x_{n-1})) \\ &\times P(t_{n-1}, x_{n-1}; dt_n, dx_n)\, p_0(t_n, x_n). \end{aligned} \tag{3.4}$$

Condition (3.2) is valid if the Radon–Nikodym derivative

$$\alpha(t, x; u, y) := \frac{\chi_{[t,T]}(u)\, c(u, y)\, P_{tr}(t, x, u, dy)\, du}{(1 - p_0(t, x))\, P(t, x; du, dy)} \qquad \text{exists,} \tag{3.5}$$

for any $(t, x) \in [0, T] \times R^d$, and

$$p_0(t, x) > 0 \qquad \text{for any} \quad (t, x) \in [0, T] \times R^d. \tag{3.6}$$

In this case the estimator (3.3) takes the form

$$\begin{aligned}\xi_1((0, 0; t_1, x_1; \ldots; t_n, x_n)) &= \alpha(0, 0; t_1, x_1) \\ &\times \cdots \times \alpha(t_{n-1}, x_{n-1}; t_n, x_n)/p_0(t_n, x_n). \end{aligned} \tag{3.7}$$

The estimator (3.7) is well known from the usual Monte Carlo scheme for solving linear integral equations. This scheme is due to von Neumann and Ulam and was used originally for solving large systems of linear algebraic equations. Later it has been applied to neutron transport problems and in other fields (cf., e.g., [11, Chap. 6], or [16]).

We obtain from Proposition 2(i), $k = 1$, that $I(t, x)$ satisfies the integral equation

$$I(t, x) = 1 + \int_t^T \int_{R^d} I(u, y)\, c(u, y)\, P_{\text{tr}}(t, x, u, dy)\, du, \tag{3.8}$$

for any $(t, x) \in [0, T] \times R^d$ such that $I_a(t, x) < \infty$. Further, Proposition 1(ii) yields that $I(t, x)$ is the pointwise limit of the successive approximations for Eq. (3.8). Similar facts were used in the classical papers by Kac [17, 18] and Cameron [7] to establish the Feynman–Kac formula (1.1). The paper [7] even contains the suggestion to use such series representations for the numerical evaluation of Wiener integrals, with a possible application of the Monte Carlo method. However, further development of numerical methods for functional integrals went on into other directions.

We did not use the connection between the functional integrals (2.1) and Eq. (3.8) explicitly in order to keep our assumptions as weak as possible. Namely, we avoided assumptions concerning the norm of the integral operator in (3.8), which are usually made in the von Neumann–Ulam scheme. They would be difficult to satisfy, if the function c is unbounded.

In simulation studies (Sect. 5) we will also use another unbiased estimator known from the von Neumann–Ulam scheme. Let μ be given via (3.4), and let p_0 and P be such that (3.5) and (3.6) hold. Then, the second estimator is defined on (Y, μ) via the formula

$$\xi_2((0, 0; t_1, x_1; \ldots; t_n, x_n)) = 1 + \sum_{k=1}^{n} \alpha(0, 0; t_1, x_1) \times \cdots \times \alpha(t_{k-1}, x_{k-1}; t_k, x_k). \tag{3.9}$$

4. Variance Reduction

A major problem in the Monte Carlo integration theory is the construction of estimators with a smallest possible variance. Many general variance reduction techniques are known (cf. [11, Chap. 4], or [24, Chap. 4]). In the case of the estimators (3.3) variance reduction can be performed by an appropriate choice of the probability measure μ. This procedure is known as "importance sampling." In Proposition 3(iv) the optimal probability measure is given. Fortunately, it can be generated by a Markov chain.

Proposition 4. *Consider a Markov chain with the state space* $\{(t, x) \in [0, T] \times R^d : I_a(t, x) < \infty\}$ *starting in* $(0, 0)$. *Let the transition parameters be*

$$p_0(t, x) = 1/I_a(t, x),$$

$$P(t, x; du, dy) = \frac{\chi_{[t,T]}(u)\, I_a(u, y)\, |c(u, y)|\, P_{tr}(t, x, u, dy)\, du}{I_a(t, x) - 1}. \tag{4.1}$$

Then the chain generates the probability measure $|v|/I_a(0, 0)$ *on* Y.

Proof. Substitution of the parameters (4.1) into (3.4) shows that μ has the expected form on

$$\bigcup_{n=0}^{\infty} \{(t_0, x_0; \ldots; t_n, x_n) \in Y_n : I_a(t_k, x_k) < \infty, k = 0, 1, \ldots, n\}.$$

But the complement of this set has the $|v|$-measure zero according to Proposition 2(iii).

However, the optimal parameters are available only in simple cases, in which $I_a(t, x)$ is known explicitly. Consider the case

$$|c(t, x)| = \gamma > 0 \qquad \text{for any} \quad (t, x) \in [0, T] \times R^d. \tag{4.2}$$

Then $I_a(t, x) = \exp(\gamma(T-t))$, and we can write the optimal parameters (4.1),

$$p_0(t, x) = \exp(-\gamma(T-t)), \tag{4.3}$$

$$P(t, x; du, dy) = \frac{\chi_{[t, T]}(u)\, \gamma \exp(\gamma(T-u))}{\exp(\gamma(T-t)) - 1} P_{\mathrm{tr}}(t, x, u, dy)\, du. \tag{4.4}$$

The mean length of the trajectories of the corresponding Markov chain can be computed to be γT. Note that the functional integral $I(0, 0)$ itself is unknown in the case (4.2), if the function c takes both values γ and $-\gamma$.

Now we consider the parameters (4.3), (4.4) in the general case. We find

$$\alpha(t, x; u, y) = \exp(\gamma(u-t))\, \gamma^{-1}\, c(u, y).$$

Therefore, the estimator (3.7) takes the simple form

$$\xi_1((0, 0; t_1, x_1; \ldots; t_n, x_n)) = \exp(\gamma T)\, \gamma^{-n} c(t_1, x_1) \times \cdots \times c(t_n, x_n). \tag{4.5}$$

The second moment of the estimator (4.5) can be computed via Proposition 3(i). We obtain

$$E\xi_1^2 = \exp(\gamma T)\, E \exp\left(\gamma^{-1} \int_0^T c^2(u, X(u))\, du\right)$$

and are able to give a priori estimates for the statistical error.

The theoretical expressions (4.1) for the optimal parameters can be used as an orientation for choosing the parameters of the Markov chain. The shape of the function c has to be taken into account, some information about the behaviour of I_a can be used, etc. Illustrative examples will follow in Section 5.

However, "importance sampling" can be considered as an appropriate variance reduction technique only if the optimal variance is zero. This holds if the function c is positive. However, in the case $c(t, x) = -\gamma < 0$, for any $(t, x) \in [0, T] \times R^d$,

we know from Proposition 3(iv) that the optimal variance equals $\exp(2\gamma T) - \exp(-2\gamma T)$. This expression can be arbitraryly large. Hence, "importance sampling" fails and other variance reduction techniques are needed. In Section 5 (Example 1) we use the following simple procedure. The estimators are constructed for the auxiliary function

$$c_1(t, x) := c(t, x) + c_0, \tag{4.6}$$

and then multiplied by the factor $\exp(-c_0 T)$ in order to correspond to the original integral. For a positive function c_1, the optimal variance available by means of "importance sampling" becomes zero again.

5. Simulation Studies

In this section we present a sample of results of a computational experiment performed with our method. The estimators (3.7) and (3.9) are used for the evaluation of three functional integrals of the form (1.2). In each case, various parameters p_0 and P are chosen in order to reduce the variance of the estimators and, by this, the statistical error made in the computation. Beside (4.3) and (4.4), we use the function

$$\exp(-(\gamma + x^2)(T-t)), \qquad \gamma > 0, \tag{5.1}$$

as a probability of absorption and the appropriately normalized function

$$|c(u, y)|\, P_u(t, x, u, dy)\, du \tag{5.2}$$

as a transition function of the Markov chain. In the first example, we use the shift procedure with the parameter c_0 mentioned at the end of Section 4 (cf. (4.6)).

First we provide some technical details necessary for understanding the numerical results. The number of independent trajectories (ω_j) $(j = 1, \ldots, n)$ of the Markov chain is $n = 5000$ in all examples. The empirical means

$$\eta_i := \frac{1}{n} \sum_{j=1}^{n} \xi_i(\omega_j), \qquad i = 1, 2,$$

are calculated simultaneously for the estimators ξ_1 and ξ_2 on the same trajectories.

Confidence intervals are constructed in order to measure the statistical error made in the evaluation of the functional integral by the empirical means η_i. The length of the confidence intervals is computed via the formulas $\delta_i := \lambda(\varepsilon)(s_i^2/n)^{1/2}$, $i = 1, 2$, where $s_i^2 := (1/n)\sum_{j=1}^{n} \xi_i^2(\omega_j) - \eta_i^2$ are the empirical variances of the estimators and $\lambda(\varepsilon)$, $\varepsilon \in (0, 1)$, is the solution of the equation

$$(2/\pi)^{1/2} \int_{\lambda(\varepsilon)}^{\infty} \exp(-x^2/2)\, dx = \varepsilon.$$

TABLE I

c_0	p_0	P	γ	Confidence intervals	l
0.0	(4.3)	(4.4)	1.0	0.609 +/− 0.049 0.630 +/− 0.018	1.0
0.5	(4.3)	(4.4)	0.5	0.643 +/− 0.018 0.633 +/− 0.013	0.5
1.0	(4.3)	(5.2)	1.0	0.639 +/− 0.008 0.645 +/− 0.008	0.9

The confidence level ε equals 0.1 in our tables, with $\lambda(0.1) = 1.6449$, but the results can easily be transformed to other confidence levels.

The efficiency of Monte Carlo algorithms depends not only on the variance of the estimators, but also on the mean length of the trajectories of the Markov chain. The empirical mean length l is also given in the tables.

First we consider the functional integral $E\exp(-\int_0^1 \chi_{[0,\infty)}(w(s))\,ds)$, where χ denotes the indicator function and w is the Wiener process having the transition density

$$p_{\mathrm{tr}}(t, x, u, y) = (\pi(u-t))^{-1/2}\exp(-(y-x)^2/(u-t)),$$

$0 \leqslant t < u \leqslant T$, $x, y \in R$. Its analytically calculated value is approximately 0.645 (cf. [18, 19]). Our results are shown in Table I, where the column "confidence intervals" contains the values $\eta_i +/- \delta_i$, $i = 1, 2$, corresponding to the estimators ξ_1 and ξ_2, respectively.

As a second example, we consider the functional integral $E\exp(\int_0^1 w(s)^2\,ds) = 1.360$ (cf. [19, 25]).

Finally, we consider the functional integral $E\exp(\int_0^1 w_{\mathrm{b}}(s)^2\,ds) = 1.197$, where w_{b} is the Brownian bridge with the transition density

$$p_{\mathrm{tr}}(t, x, u, y) = (2\pi a)^{-1/2}\exp(-(y - x(T-u)/(T-t))^2/2a),$$
$$a = (T-u)(u-t)/(T-t), \qquad 0 \leqslant t < u < T,\ x, y \in R.$$

TABLE II

p_0	P	γ	Confidence intervals	l
(4.3)	(4.4)	1.0	1.361 +/− 0.051 1.371 +/− 0.052	1.0
(4.3)	(5.2)	1.0	1.338 +/− 0.025 1.346 +/− 0.011	0.8
(5.1)	(5.2)	0.3	1.357 +/− 0.010 1.340 +/− 0.018	0.3

TABLE III

p_0	P	γ	Confidence intervals	l
(4.3)	(4.4)	1.0	1.190 +/− 0.030 1.202 +/− 0.015	1.0
(4.3)	(5.2)	1.0	1.194 +/− 0.027 1.196 +/− 0.004	1.0

Considerable variance reduction has been achieved in all examples, although only very simple techniques were used. Note that the estimator ξ_1 with the second set of parameters in Table I has the minimal variance possible within the pure importance-sampling procedure. The third set of parameters shows that the shift procedure (4.6) can make "importance sampling" more efficient.

It is to be mentioned that various estimators should be compared first with respect to their confidence intervals and not with respect to the concrete values of their empirical means. We calculated, for instance, the estimators with the third set of parameters in Table I with another sample of 5000 independent trajectories and obtained the confidence intervals 0.643 +/− 0.008 and 0.639 +/− 0.008, respectively.

6. Concluding Remarks

The main advantage of our unbiased estimation scheme for functional integrals is that no systematic error is involved. Therefore the error analysis becomes much more easy, and can be performed by means of confidence intervals during the process of generation of independent samples of the estimator.

In many cases unbiased estimators seem to be even more efficient in the following sense. Usual biased estimators correspond to the evaluation of an integral of finite but relatively high dimension. The unbiased estimation scheme can be interpreted as the evaluation of integrals of finite dimension, which is chosen at random by means of the length of the basic Markov chain. Then, the mean dimension equals the mean length of the trajectories of the Markov chain, which is rather small in our examples.

Condition (3.1) is essential for our algorithm. It excludes such functions c, which tend to $-\infty$ too fast. However, for positive functions c, we need only the existence of the object to be evaluated. In particular, no regularity requirement on c is necessary. The algorithm works for arbitrary Markov processes with an explicitly given transition function. In particular, inhomogeneous processes and processes with jump components are covered.

Functional integrals of the form

$$E \exp\left(\int_0^T c(u, X(u))\, du\right) g(X(T))$$

or

$$E\int_0^T \exp\left(\int_0^v c(u, X(u))\,du\right) f(v, X(v))\,dv$$

appearing in more general Feynman–Kac formulas can be handled in a quite similar way as the integrals (1.2). The unbiased evaluation of integrals of the form

$$E\,G\left(\int_0^T \cdots \int_0^T c(u_1,..., u_n, X(u_1),..., X(u_n))\,du_1 \cdots du_n\right),$$

$n \geqslant 1$, G-analytic, is possible but would need more sophisticated variance reduction techniques.

We carried out only the importance-sampling procedure in the framework of the von Neumann–Ulam scheme. In order to develop other variance reduction techniques it would be useful to consider this scheme as a part of the general Monte Carlo estimation theory for generalized principal values of integrals. In this connection we refer to [27, 28] and [12, Chap. 6].

Numerical methods for stochastic Wiener integrals, where the function c in (1.2) depends on random influences, have been introduced in [3] in connection with scattering in random media.The evaluation of such functional integrals seems to be a very promising field of application of our unbiased estimation scheme.

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